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The trapping transition in dynamic (invasion) and static percolation

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Abstract. For standard 2D bond percolation, the size of regions trapped by the infinite occupied cluster at bond density p is studied by Monte Carlo simulations. It is known that there is a transition at some density $p_t > 1/2$ which determines the fractal behaviour of invasion percolation with trapping. The numerical results are that $p_t \approx 0.520$ and the critical exponents are those ($\gamma = 43/18$ and $\nu = 4/3$) of the usual percolation transition at $p_c = 1/2$. Thus invasion percolation with trapping does not appear to belong to a new universality class.

In this paper we report the results of a Monte Carlo study of the 'trapping transition' in standard bond percolation on the 2D square lattice. As explained below, this transition occurs as the mean size of regions trapped by the infinite occupied cluster diverges when the bond density p is lowered to a critical value p_t within the percolating phase. One reason for interest in this transition is its relation to invasion percolation (with trapping).

Invasion percolation is a dynamic growth model originally invented for the study of two-fluid interfaces in a porous medium (Lenormand and Bories 1980, Chandler *et al* 1982). Random numbers between 0 and 1 are assigned independently to the bonds of the lattice. The displacing fluid (which is initially located, say, at the origin) chooses at each time step the path of least resistance by invading that bond on the boundary of the currently invaded region with the smallest random number. The large-time fractal behaviour of this version of invasion percolation (i.e. without trapping) has been shown (Wilkinson and Willemsen 1983, Willemsen 1984, Nickel and Wilkinson 1983, Wilkinson and Barsony 1984, Chayes *et al* 1985) both numerically and analytically to be controlled by the usual critical point $p = p_c$ of standard (static) percolation. A more interesting version of invasion percolation adds a trapping rule, which forbids invasion into any region which is completely trapped by invaded bonds. A region becomes trapped when any path to infinity must pass through some already invaded bond. The trapping rule represents physically the incompressibility of the displaced fluid.

In Wilkinson and Willemsen (1983) and Willemsen (1984), 2D and 3D simulations of invasion percolation with trapping were performed in which the invasion was continued until all bonds in the finite simulation volume were either invaded or trapped. It was thought that, in this situation, the random numbers of invaded bonds (asymptotically) include values up to but not beyond $1-p_c$—compared with p_c for invasion percolation without trapping. The numerical results then led to two conclusions: first, that the 2D situation was degenerate since $p_c = 1 - p_c$ and, second, that the 3D fractal behaviour, in spite of being related to the critical point $1 - p_c$ for percolating vacant bonds, corresponded to a new universality class different from the usual percolation phase transition. This was of considerable interest since it supported the notion that dynamical models could exhibit novel critical behaviour (Kadanoff 1989).

However, the first conclusion was disproved and the second conclusion cast in some doubt by subsequent theoretical work of Chayes *et al* (unpublished). They showed that bonds with random numbers up to the trapping transition density p_t are invaded, and proved that at least for 2D, $p_t > 1 - p_c$. Their work indicates that for any dimension, no new universality class occurs for invasion percolation with trapping if it does not occur for the static trapping transition at p_t, and, further, that in this respect 2D is not degenerate. The main purpose of this paper is then to numerically investigate whether the 2D trapping transition does or does not represent a new universality class. If it does not, there is no reason to believe that a new universality class will appear for any higher dimension. We note that the study of trapping in a static percolation model has a major advantage over its dynamic counterpart—it is much simpler to simulate.

To define the trapping tranisition more pecisley, start with a standard bond percolation model on a d-dimensional cubic lattice. At any bond density p, 'remove' from the lattice all those bonds (but not the sites) which are a part of the infinite occupied cluster. A trap is then defined as a connected component of what remains of the lattice. For p close to 1, all traps will be finite. As p is lowered, the trapping transition occurs at the value p_t below which there is an infinite trap. Below p_t, there is a non-zero probability for the existence of a path from the origin to infinity consisting entirely of vacant bonds and bonds belonging to finite occupied clusters; in this region the infinite occupied cluster does not trap all the rest of the lattice.

For any dimension d, it is clear that there will be an infinite trap for all $p < 1 - p_c$, since then vacant bonds percolate; thus $p_t \geqslant 1 - p_c$. It has been proved that $p_t > 1 - p_c$ first for $d = 2$ (Chayes *et al*, unpublished) and then for general d (Aizenman and Grimmett 1989). Another quantity of interest is $\chi_t(p)$, the mean number of sites in the trap containing the origin. The main questions which we have addressed for $d = 2$ are the numerical value of p_t (by how much does it exceed $1 - p_c = 1/2$?) and the critical behaviour of the trapping transition (is it different from the usual percolation transition?). The critical exponent γ, for example, is defined by $\chi_t(p) \sim (p - p_t)^{-\gamma}$ as $p \downarrow p_t$; the analogous exponent for the usual 2D percolation transition has the exact value 43/18 (den Nijs 1979, Nienhuis *et al* 1980, Pearson 1980). By analysing numerical data for $\chi_t(p)$ at different lattice sizes, we are able to test whether both γ and the correlation length exponent ν are the same for the trapping transition as for the usual percolation transition.

We have studied trapping via 2D numerical simulations on a square lattice with L sites to a side. All simulations were performed on a Sun 3 workstation. The simulation procedure is as follows. Given a value of p, first identify the occupied bonds as those random numbers below p. Next, identify the set I, which consists of all bonds belonging to occupied clusters which reach the boundary of the lattice. Finally identify T, the collection of all those connected components of bonds in the complement of I which intersect a square R centred on the origin with $L/4$ sites to a side. I represents (in fact overestimates) the set of bonds in the $L \times L$ lattice from the infinite occupied cluster, while T represents the set of traps which come 'close' to the origin. We use T rather than just the trap of the origin in order to improve the statistics of trap sizes.

Studying invasion percolation with trapping indirectly through the trapping transition has several notable numerical advantages. We work in a completely static model, which can be easily simulated whereas invasion percolation itself is rather more difficult to simulate. More importantly, searching for trapped regions in invasion percolation is a time consuming task, both because it must be carried out at every time step, and because inclusion in a trapped region is not a local property. In our algorithm, finding whether a trap reaches the boundary of the lattice is a trivial by-product of counting its size.

We conducted 2D simulations for $L = 50$, 100, 200, and 350. The range of p values used depended on the size of the lattice, and the increment between p values was generally 0.001. The overall range of p values across all lattice sizes was from 0.51 to 0.60. The main quantity which we measured was the mean trap size (for traps in T) $\chi_t(p, L)$. The appendix lists the measured values for this quantity along with error brackets (as explained in detail below). A graph showing the dependence of the mean trap size on L and p is given in figure 1. To analyse our data we used finite-size scaling methods (see, e.g., Barber 1983). Given L and the critical exponents γ and ν (for the infinite lattice), the finite-size scaling ansatz is that

$$\chi_t(p, L) = L^{\gamma/\nu} f((p - p_t)L^{1/\nu})$$

for some fixed scaling function f (independent of L). Taking the numerical values of $\chi_t(p, L)$ for a given L and varying p, one can invert this equation to obtain a numerical estimate of the function f, which we denote by f_L. Of course f_L depends on the choice of p_t, γ and ν. We plotted the functions f_L for all four lattice sizes on a single graph for various values of p_t, γ and ν. When the values of p_t, γ and ν are 'correct,' all the data should lie on one curve. To decide whether the trapping transition is in the same

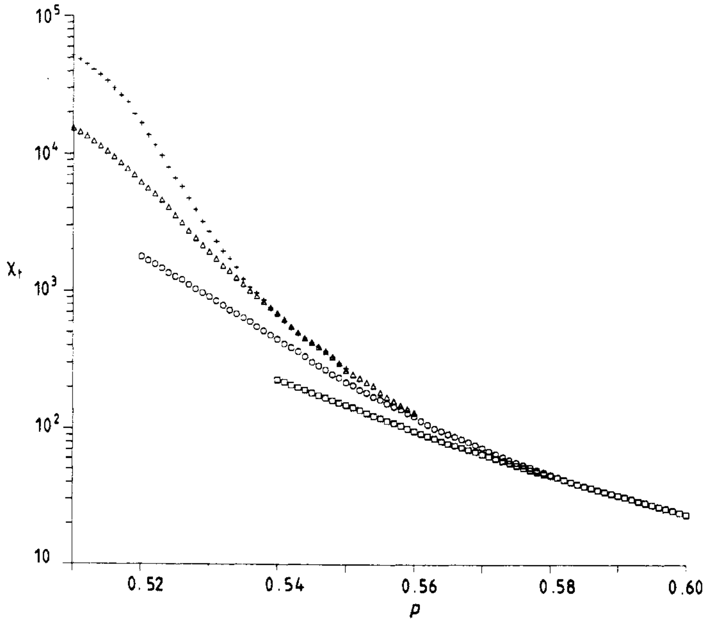

Figure 1. A graph of χ_t, the mean trap size, against p, the density of occupied bonds, for four different lattice sizes L. The square data points are for $L = 50$, the octagons for $L = 100$, the triangles for $L = 200$ and the crosses for $L = 350$. The vertical scale is logarithmic.

universality class as standard percolation we first set $\gamma = 43/18$ and $\nu = 4/3$ and graphed the f_L for a range of p_t values. The best fit occurred with $p_t = 0.520$ (see figure 2). As we varied p_t, γ and ν from these values, the fit deteriorated. See figures 3, 4 and 5 for examples. The data are clearly consistent with the 'null hypothesis' that the trapping transition is in the same universality class as the standard percolation transition.

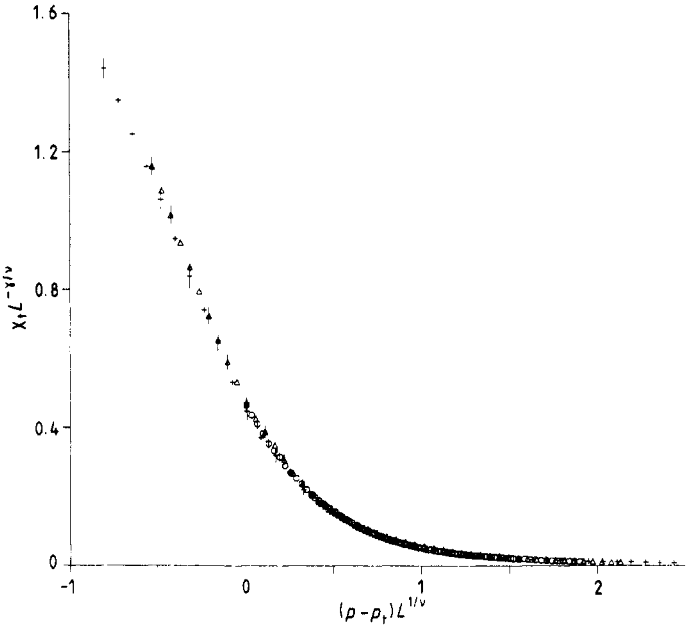

Figure 2. A graph of $\chi_t L^{-\gamma/\nu}$ against $(p-p_t)L^{1/\nu}$ for $L = 50$ (squares), 100 (octagons), 200 (triangles) and 350 (crosses). Here $\gamma = 43/18 \approx 2.39$, $\nu = 4/3 \approx 1.33$ and $p_t = 0.520$. Vertical error bars of $\pm$ one standard deviation are included for every other data point.

For a given p and L, the numerical estimates of $\chi_t(p, L)$ are averages over different simulation runs (denoted ω) of $\bar{\chi}(\omega)$, the weighted average of the sizes of traps in T:

$$\bar{\chi}(\omega) = (L/4)^{-2} \sum_{x \in \mathbb{R}} \mathscr{T}(x, \omega)$$

where for the run ω, $\mathscr{T}(x, \omega)$ is the number of sites in the trap containing x when the trap belongs to T (and otherwise $\mathscr{T}(x, \omega) = 0$). For a given run ω, these trap sizes are not independent. However, the $\bar{\chi}$ are independent for different runs; thus we estimate error brackets for $\chi_t(p, L)$ by taking the usual sample standard deviation for the data consisting of the different $\bar{\chi}(\omega)$. The number of runs varied from 100 to 1000. Sample error brackets appear in figure 2; more complete information is given in the appendix.

Another feature of our simulation procedure allowed us to use the same random numbers (used in determining bond occupation) for many different values of p. To do this we simply kept track of the set I, and added bonds to it as we increased the value of p. This procedure forces the trap sizes to be monotonic in p for each run. While this procedure should not decrease the statistical error for a single $\chi_t(p, L)$ or f_L value, it certainly makes these values for different p dependent and apparently smoothes the data. Perhaps this helps explain why the fit between the four f_L (which are independent) in figure 2 seems to somewhat exceed the accuracy suggested by the error brackets.

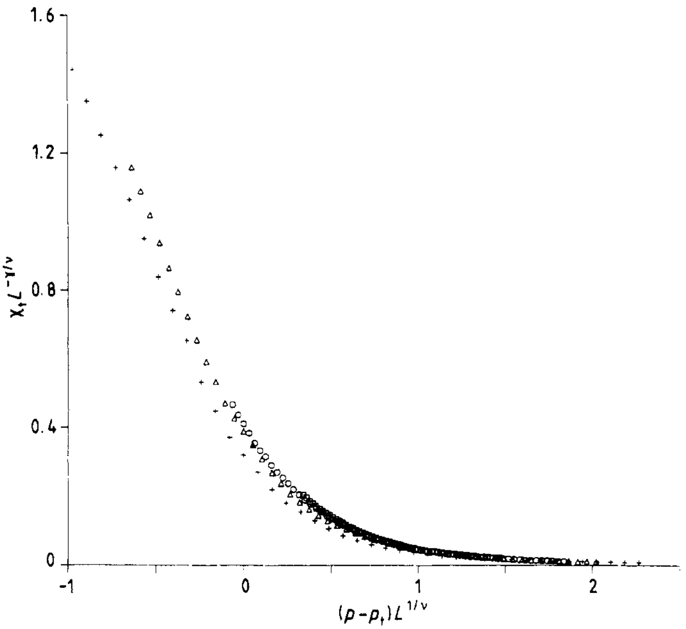

Figure 3. The same graph as figure 2 except that p_t has been changed from 0.520 to 0.522.

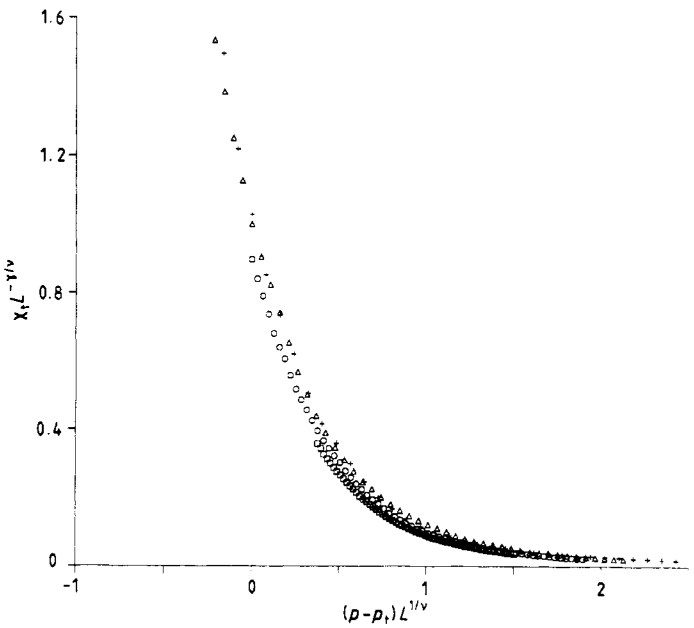

Figure 4. The same graph as figure 2 except that γ has been changed from 2.39 to 2.20.

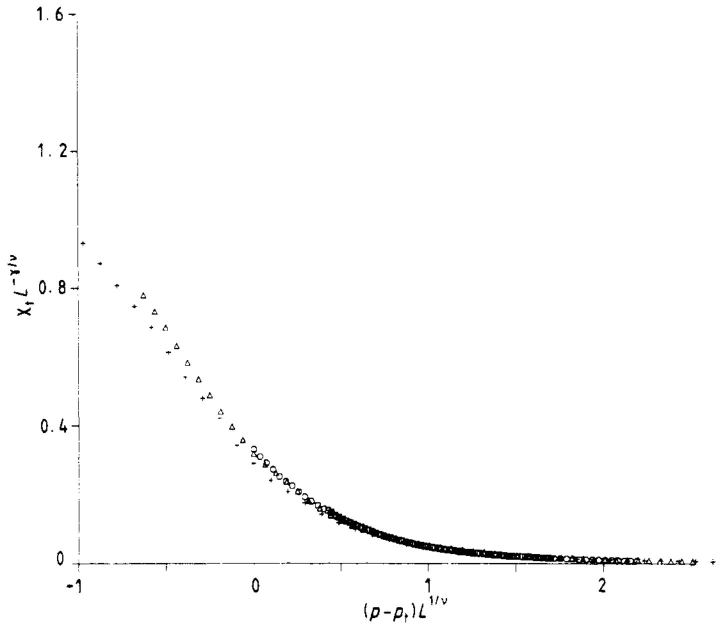

Figure 5. The same graph as figure 2 except that ν has been changed from 1.34 to 1.28.

In conclusion, we have presented evidence that invasion percolation with trapping is in the same universality class as the regular percolation transition. This evidence comes from studying numerically the static percolation analogue, the trapping transition, and showing, at least in 2D, that it is in the same universality class as regular percolation.

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Appendix

	$\chi_t(p, L) \pm$ one standard deviation			
p	$L=50$	$L=100$	$L=200$	$L=350$
0.510			15 354.0 ± 349.9	52 113.0 ± 1044.0
0.511			14 422.0 ± 342.5	48 780.0 ± 1060.0
0.512			13 504.0 ± 338.0	45 220.0 ± 1107.0
0.513			12 439.0 ± 344.2	41 807.0 ± 1106.0

	$\chi_t(p, L) \pm$ one standard deviation			
p	$L = 50$	$L = 100$	$L = 200$	$L = 350$
0.514			11 474.0 ± 335.6	38 394.0 ± 1176.0
0.515			10 535.0 ± 331.2	34 295.0 ± 1260.0
0.516			9 608.8 ± 319.8	30 288.0 ± 1234.0
0.517			8 669.3 ± 313.5	26 795.0 ± 1215.0
0.518			7 826.3 ± 307.0	23 586.0 ± 1129.0
0.519			7 060.2 ± 284.9	19 203.0 ± 1065.0
0.520		1787.1 ± 65.40	6 251.7 ± 262.1	16 187.0 ± 943.3
0.521		1674.5 ± 63.53	5 658.7 ± 245.0	13 429.0 ± 846.0
0.522		1576.1 ± 61.78	5 155.6 ± 228.0	11 568.0 ± 787.0
0.523		1469.5 ± 59.82	4 639.3 ± 213.3	9 803.8 ± 711.3
0.524		1358.4 ± 56.00	4 098.3 ± 202.2	7 999.6 ± 588.7
0.525		1279.7 ± 53.92	3 570.5 ± 182.2	6 564.0 ± 521.6
0.526		1211.0 ± 51.89	3 155.6 ± 167.3	5 644.2 ± 470.0
0.527		1116.6 ± 47.27	2 754.2 ± 151.7	4 710.2 ± 380.0
0.528		1036.6 ± 43.57	2 440.4 ± 136.3	3 880.4 ± 302.4
0.529		975.29 ± 41.91	2 165.5 ± 120.6	3 153.4 ± 233.6
0.530		911.89 ± 39.99	1 938.7 ± 106.1	2 632.1 ± 197.4
0.531		849.14 ± 38.69	1 729.3 ± 97.69	2 185.8 ± 169.2
0.532		788.76 ± 36.71	1 537.4 ± 90.38	1 829.0 ± 127.9
0.533		730.37 ± 34.38	1 411.3 ± 86.16	1 613.5 ± 117.7
0.534		685.01 ± 32.47	1 259.8 ± 78.82	1 407.1 ± 106.4
0.535		641.47 ± 31.54	1 128.8 ± 70.37	1 175.8 ± 79.43
0.536		602.44 ± 30.51	1 015.0 ± 62.58	1 028.3 ± 67.73
0.537		551.71 ± 27.31	923.98 ± 57.65	931.05 ± 59.92
0.538		514.22 ± 26.39	843.13 ± 53.79	837.34 ± 56.46
0.539		478.07 ± 24.70	762.97 ± 48.30	741.63 ± 47.65
0.540	227.11 ± 6.218	446.90 ± 22.80	693.72 ± 45.21	668.01 ± 42.02
0.541	218.53 ± 6.009	413.37 ± 21.51	626.04 ± 42.10	621.04 ± 40.47
0.542	207.39 ± 5.610	387.87 ± 20.47	552.59 ± 36.36	550.92 ± 35.95
0.543	199.47 ± 5.470	364.19 ± 19.36	501.07 ± 34.45	501.60 ± 33.14
0.544	190.98 ± 5.300	335.00 ± 17.64	457.48 ± 31.39	460.98 ± 32.04
0.545	182.32 ± 5.100	305.42 ± 15.20	425.14 ± 30.10	432.58 ± 30.71
0.546	174.87 ± 4.911	287.84 ± 14.23	394.34 ± 28.07	405.12 ± 28.91
0.547	168.47 ± 4.761	269.78 ± 13.54	366.42 ± 27.16	375.74 ± 27.49
0.548	161.54 ± 4.577	249.68 ± 12.25	334.19 ± 23.50	344.57 ± 24.35
0.549	155.15 ± 4.415	236.08 ± 11.70	300.00 ± 20.26	304.41 ± 16.46
0.550	148.54 ± 4.300	216.92 ± 10.79	266.26 ± 15.71	282.84 ± 14.62
0.551	143.11 ± 4.174	206.48 ± 10.45	249.17 ± 14.36	
0.552	136.82 ± 4.005	191.86 ± 9.041	233.98 ± 13.23	
0.553	129.82 ± 3.790	183.02 ± 8.872	217.83 ± 11.64	
0.554	124.24 ± 3.666	170.55 ± 7.970	205.99 ± 11.13	
0.555	120.00 ± 3.604	162.92 ± 7.688	184.88 ± 9.139	
0.556	114.71 ± 3.464	153.26 ± 7.216	171.48 ± 8.263	
0.557	110.40 ± 3.371	145.50 ± 6.834	160.06 ± 7.746	
0.558	105.84 ± 3.289	139.13 ± 6.688	148.21 ± 7.184	
0.559	100.88 ± 3.082	130.41 ± 6.321	140.09 ± 6.709	
0.560	96.721 ± 2.978	124.14 ± 6.112	132.12 ± 6.196	
0.561	92.607 ± 2.858	114.82 ± 5.333		
0.562	88.885 ± 2.750	107.94 ± 4.953		
0.563	85.467 ± 2.658	101.26 ± 4.628		
0.564	81.580 ± 2.561	97.189 ± 4.476		
0.565	78.282 ± 2.338	92.112 ± 4.196		
0.566	75.869 ± 2.279	88.415 ± 4.029		
0.567	72.705 ± 2.196	84.597 ± 3.888		

	$\chi_t(p, L) \pm$ one standard deviation			
p	$L=50$	$L=100$	$L=200$	$L=350$
0.568	69.939 ± 2.116	80.492 ± 3.752		
0.569	67.443 ± 2.066	76.349 ± 3.351		
0.570	64.633 ± 1.883	72.681 ± 3.081		
0.571	62.339 ± 1.819	68.268 ± 2.767		
0.572	60.075 ± 1.775	65.145 ± 2.592		
0.573	57.763 ± 1.698	61.726 ± 2.433		
0.574	55.337 ± 1.641	58.452 ± 2.253		
0.575	53.238 ± 1.570	55.842 ± 2.104		
0.576	51.220 ± 1.491	53.703 ± 2.065		
0.577	49.410 ± 1.418	51.638 ± 1.975		
0.578	47.839 ± 1.380	49.367 ± 1.854		
0.579	46.336 ± 1.344	47.313 ± 1.779		
0.580	44.952 ± 1.282	45.509 ± 1.707		
0.581	43.611 ± 1.261			
0.582	41.767 ± 1.188			
0.583	40.191 ± 1.123			
0.584	38.832 ± 1.078			
0.585	37.504 ± 1.038			
0.586	36.538 ± 1.021			
0.587	35.269 ± 0.986			
0.588	34.217 ± 0.954			
0.589	33.121 ± 0.931			
0.590	32.115 ± 0.896			
0.591	31.220 ± 0.879			
0.592	30.243 ± 0.840			
0.593	29.217 ± 0.801			
0.594	28.425 ± 0.783			
0.595	27.617 ± 0.760			
0.596	26.685 ± 0.742			
0.597	25.804 ± 0.713			
0.598	24.971 ± 0.681			
0.599	24.130 ± 0.646			
0.600	23.502 ± 0.625			

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